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COLLISION-FREE MOTION IN SYSTEMS WITH NON-RETAINING CONSTRAINTS†

A. P. IVANOV

Moscow

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The properties of collision-free periodic motions in a system with a non-retaining constraint are studied. In motions of this kind the intervals in which the constraint is in a strained or slackened state alternate, and moreover the transition of the constraint to a strained state does not involve collisions. Conditions are derived for the existence of collision-free motions, and their relationship to singularities of differentiable mappings is ascertained. The "attracting" properties of collision-free motions are established: under certain conditions they are analogous to semistable limit cycles. The results are illustrated by examples: a body on an elastic leg [1] and a link-up of bodies in a satellite orbit [2].

THE PRACTICAL interest in collision-free motions is due to the possibility of "flights" of strained states of the constraints between different parts, without the undesirable effects associated with collisions, such as overloading, vibrations, energy loss, etc. [1]. However, the general properties of such motions have yet to be investigated; the few results available at present (such as [2]) are far from satisfactory.

1. DIFFERENT TYPES OF MOTION AND THEIR DESCRIPTION

Let $\mathbf{q} = (q_1, \ldots, q_n)$ be the generalized coordinates of a mechanical system, $q_1 \geqslant 0$ a non-retaining constraint, $L = L(\mathbf{q}, \mathbf{q}^{\cdot})$ the Lagrangian, $\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{q}^{\cdot}) = (Q_1, \ldots, Q_n)$ generalized forces. We assume that the functions Q_i are continuously differentiable and that L is twice continuously differentiable in some domain G of the phase space R^{2n}; the non-retaining constraint is assumed to be ideal. On the assumption that there are no collisions with the constraint, the Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_1^{\cdot}}\right) - \frac{\partial L}{\partial q_1} = Q_1 + R, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial q_j^{\cdot}}\right) - \frac{\partial L}{\partial q_j} = Q_j \quad (j = 2, \ldots, n) \tag{1.1}$$

where $R \geqslant 0$ is the reaction of the constraint.

System (1.1) contains $n+1$ unknowns: $R, q_1^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot}$, so that for unique solvability we need an additional relation among the unknowns. A suitable relation for R and $q_1^{\cdot\cdot}$ in the case of an ideal constraint is [3]

$$R \neq 0 \Rightarrow q_1 = q_1^{\cdot} = q_1^{\cdot\cdot} = 0 \tag{1.2}$$

There are two basic situations in which system (1.1) is solvable:

Case 1. The non-retaining constraint is slackened, $q_1 \geqslant 0$; then $R = 0$.

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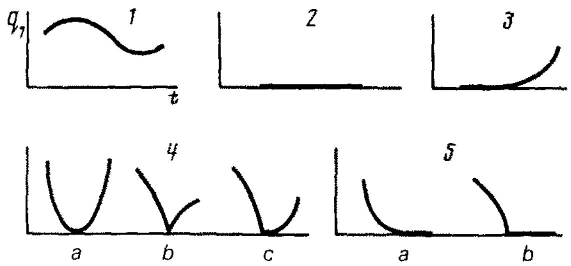

FIG. 1.

Case 2. The constraint is strained, $R>0$; then by (1.2) $q_1^{\cdot\cdot}=0$. In that case we can first determine $q_2^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot}$ from the second group of Eqs (1.1) and then substitute the results into the first equation to get R. Since the constraint is unilateral, only values $R\geqslant 0$ are admissible. If the assumption $q_1^{\cdot\cdot}=0$ implies that $R<0$, it is not legitimate; it must then be assumed that $R=0$, $q_1^{\cdot\cdot}>0$ and the constraint is slackened.

Certain motions of system (1.1) are difficult to investigate: these are motions that involve both basic mode cases 1 and 2. When the constraint switches from one state to the other at some time $t=t^*$, the following cases may arise (Fig. 1).

Case 3. The constraint is strained for $t\leqslant t^*$, slackened for $t>t^*$.

Case 4. The constraint is slackened for $t<t^*$ and $t>t^*$, strained at $t=t^*$.

(a) If $q_1^{\cdot}(t^*-0)=0$, the phase curve $(\mathbf{q}(t), \mathbf{q}^{\cdot}(t))$ is continuous in the phase space R^{2n} and touches the plane $q_1=0$.

(b) If $q_1^{\cdot}(t^*-0)<0$, a collision with the constraint occurs at $t=t^*$, i.e. the generalized velocities change abruptly, during an infinitesimal time interval; the change in the velocity $q_1^{\cdot}$ is described in terms of Newton's coefficient of restitution $\varkappa$:

$$q_1^{\cdot}(t^*+0)=-\varkappa q_1^{\cdot}(t^*-0), \quad 0\leqslant\varkappa\leqslant 1 \tag{1.3}$$

(c) $\varkappa=0$ in (1.3), but substitution of $q_1^{\cdot\cdot}=0$ into Eq. (1.1) at $t=t^*+0$ yields a negative reaction R.

Case 5. The constraint is strained for $t\geqslant t^*$ and slackened for $t<t^*$:

(a) If $q_1^{\cdot}(t_1-0)=0$, the phase curve is continuous and the straining of the constraint is not accompanied by a collision.

(b) If $q_1^{\cdot}(t_1-0)<0$, $\varkappa=0$ in (1.3) and substitution of $q_1^{\cdot\cdot}=0$ into (1.1) gives $R\geqslant 0$, this is the case of plastic collision.

In addition, there is another possibility: infinite alternation of strained and slackened states of the constraint in the neighbourhood of $t=t^*$, e.g. in the case of quasi-plastic collision [4]. The collision-free motions that we are going to discuss include, apart from the basic modes 1 and 2, cases 3 and 5a.

2. CONDITIONS FOR THE EXISTENCE OF COLLISION-FREE MOTIONS

Let us examine the conditions for slackening of the constraint (case 3) and collision-free straining (case 5a).

Definition 2.1. Using Eqs (1.1), we express $q_1^{\cdot\cdot}$ in terms of $\mathbf{q}, \mathbf{q}^{\cdot}$ in the case $R\equiv 0$. The locus of solutions of the equation

$$q_1^{\cdot\cdot}(\mathbf{q}, \mathbf{q}^{\cdot})=0 \tag{2.1}$$

will be called the separation surface in the phase space R^{2n}, denoted by J. The domain in which $q_1^{\cdot\cdot}>0$ will be denoted by J^+, and the domain in which the opposite inequality holds will be denoted by J^-.

Under our assumptions, the left-hand side of Eq. (2.1) is continuously differentiable. Hence J is a closed set, while J^+ and J^- are open.

Motion under the effect of a strained constraint may only occur in the domain J^- or on its boundary J. If the constraint is slackened when $t>t^*$ and the phase curve is continuous, it follows from Taylor's formula, since $q_1(t^*)=q_1^{\cdot}(t^*)=0$, that

$$0 < q_1(t) = {}^1/_2 q_1^{\cdot\cdot}(\xi)(t-t^*)^2, \quad \xi \in (t^*, t)$$

that is, the representative point at $t=\xi$ lies in J^+. Letting $t \to t^*+0$, we obtain $[\mathbf{q}(t^*), \mathbf{q}^{\cdot}(t^*)] \in J^+ \cup J$.

We have thus proved the following necessary condition for slackening of the constraint.

Theorem 2.1. If $q_1=0$ for $t \leqslant t^*$ and $q_1>0$ for $t>t^*$, then at $t=t^*$ the representative point lies on the separation surface J.

Remark. At the instant of separation, R also vanishes; since by the definition of J we have $q_1^{\cdot\cdot}=0$ whenever $R=0$, the assumption $R>0$ implies that $q_1^{\cdot\cdot}>0$, contrary to condition (1.2).

Theorem 2.2. Suppose that when $t<t^*$ the phase curve $[\mathbf{q}(t), \mathbf{q}^{\cdot}(t)]$ lies in the plane $q_1=0$ and in the domain J^- and that when $t=t^*$ it passes through a non-singular point of J, forming a non-zero angle with that surface. Then at times $t>t^*$ sufficiently close to t^* the constraint is slackened, $q_1(t)>0$.

Proof. Since $R=0$ at $t=t^*$, as pointed out above, it follows, assuming that $R\equiv 0$ for $t>t^*$, that we can continue the phase curve while preserving its continuous differentiability. By assumption, the curve intersects J and enters the domain J^+, so that such a continuation procedure yields a solution of system (1.1). By condition (1.2), there are no other continuous extensions of the solution. Since $q_1^{\cdot\cdot}>0$ in J^+ but $q_1(t^*)=q_1^{\cdot}(t^*)=0$, it follows that $q_1(t)>0$ for $t>t^*$, which it was required to prove.

Corollary. Under the assumptions of the theorem, $q_1^{\cdot\cdot\cdot}(t^*+0)>0$.

Indeed, the existence of $q_1^{\cdot\cdot\cdot}$ when $R\equiv 0$ follows from our assumptions about the form of system (1.1). In a neighbourhood of t^* with $t>t^*$, the derivative $q_1^{\cdot\cdot}$ is of the order of $t-t^*$, since the tangent vector to the phase curve makes a non-zero angle with the surface $q_1^{\cdot\cdot}=0$. Thus the second term in Taylor's formula

$$q_1^{\cdot\cdot}(t) = q_1^{\cdot\cdot}(t^*) + q_1^{\cdot\cdot\cdot}(t^*+0)(t-t^*) + o(t-t^*) \tag{2.2}$$

is positive.

Example 2.1. Let us investigate the vertical motion of a two-body system [1] consisting of a body elastically coupled to a leg (Fig. 2), under the action of the force of gravity. Let q_1, q_2 be the distances from the leg and the body to the support, m_1, m_2 their masses, k the stiffness of the spring (its mass is assumed to be negligibly small) and g the acceleration due to gravity. The Lagrangian is

$$L = {}^1/_2 m_1 q_1^{\cdot 2} + {}^1/_2 m_2 q_2^{\cdot 2} - g(m_1 q_1 + m_2 q_2) - {}^1/_2 k (q_2 - q_1 - a)^2, \quad q_1 \geqslant 0 \tag{2.3}$$

where a is the difference $q_2 - q_1$ in the unstrained state of the spring. Equations (1.1) take the form

$$q_1^{\cdot\cdot} + g - k m_1^{-1}(q_2 - q_1 - a) = R m_1^{-1}, \quad q_2^{\cdot\cdot} + g + k m_2^{-1}(q_2 - q_1 - a) = 0 \tag{2.4}$$

The equation of the surface J is obtained by isolating $q_1^{\cdot\cdot}$ in the first equation of (2.4) when $R=0$:

$$q_1^{\cdot\cdot} = k m_1^{-1}(q_2 - q_1 - a) - g = 0 \tag{2.5}$$

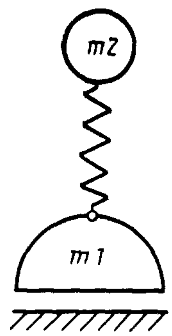

FIG. 2.

Replacing the equality in (2.5) by the inequality > or <, we obtain formulas defining J^+ and J^-, respectively (see Fig. 3a).

The motion when the constraint is strained (i.e. when the leg is in contact with the support) is described by the variables q_1, $q_2^{\cdot}$ and is conveniently pictured in the phase plane (Fig. 3a). The characteristic feature of this motion is that the mechanical energy of the system is constant:

$$h = {}^1/_2 m_2 q_2^{\cdot 2} + {}^1/_2 k (q_2 - a)^2 + m_2 g q_2 = \text{const} \tag{2.6}$$

Curves (2.6) for different h values are shown in Fig. 3(a). The system has a single equilibrium position, when $q_2 = q_2^* = a - gm_2k^{-1}$, which is the centre of a family of concentric ellipses. The only parts of the curves corresponding to real motions are those lying entirely in J^- (for example, curve 1 describes a periodic motion of the system). By Theorem 2.2, motion of the representative point along an ellipse intersecting the curve J will end on the latter; the coupling with the support will then be slackened and the phase curve will leave the plane $q_1 = 0$.

We now examine the conditions for collision-free straining of the constraint. If $q_1 > 0$ for $t > t^*$ and $q_1(t^*) = 0$, then, since in collision-free motion the phase curve is continuous, it follows that $q_1^{\cdot}(t^* - 0) \leqslant 0$.

If $q_1^{\cdot}(t^* - 0) < 0$, the system will experience a collision with the constraint, which reduces to one of the cases 4(b, c) or 5(b) described in Sec. 1.

If $q_1^{\cdot}(t^* - 0) = 0$, then $q_1^{\cdot}$ is continuous at $t = t^*$ and no collision will occur (cases 4a or 5a). Since $q_1 > 0$ when $t < t^*$ and $q_1(t^*) = q_1^{\cdot}(t^*) = 0$, it follows from Taylor's formula that

$$q_1(t) = {}^1/_2 (t - t^*)^2 q_1^{\cdot\cdot}(\xi), \quad t < \xi < t^*$$

whence we obtain $q_1^{\cdot\cdot}(\xi) > 0$. Letting $t \to t^* - 0$ in the last inequality, we obtain $q_1^{\cdot\cdot}(t^* - 0) \geqslant 0$. Strict inequality means that the constraint is slackened when $t > t^*$ (case 4a). We have thus proved the following theorem.

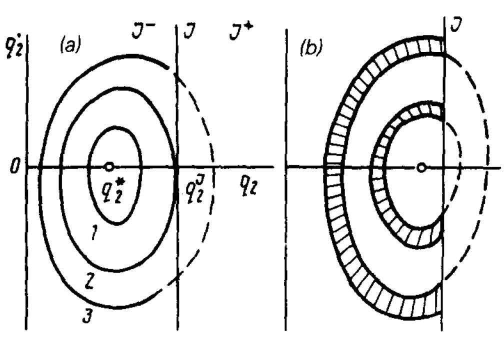

FIG. 3.

Theorem 2.3. Let $q_1>0$ for $t<t^*$ and $q_1=0$ for $t\geqslant t^*$, and assume that the phase curve is continuous. Then $q_1^{\bullet\bullet}(t^*)=0$, i.e. at the instant the constraint is strained the representative point hits the separation surface J.

We will now formulate the sufficient conditions for collision-free straining of the constraint.

Theorem 2.4. Suppose that the $t<t^*$ the phase curve lies in the domains $q_1>0$ and J^+, but at $t=t^*$ the representative point hits the intersection of the plane $q_1=q_1^\bullet=0$ and the surface J at a non-singular point of the latter, in such a way that the curve and the surface make a non-zero angle. Then $q_1=0$ when $t>t^*$ and the phase curve is continuously differentiable.

Proof. Under our assumptions, $q_1(t^*-0)=0=q_1^\bullet(t^*-0)=q_1^{\bullet\bullet}(t^*-0)$. Set $q_1\equiv 0$ for $t>t^*$ and determine the values of the variables $q_2(t),\ldots,q_n(t)$ by solving Eqs (1.1). The extended curve thus obtained is continuously differentiable, and therefore when $t>t^*$ it lies in J^-, so that $R>0$ in (1.1) and these equations are satisfied. The uniqueness of the solution follows from the fact that a change in the value of R will cause the violation of one of the conditions $q_1\geqslant 0$ or (1.2).

Corollary. Under the assumptions of the theorem, $q_1^{\bullet\bullet\bullet}(t^*-0)<0$.

The proof is analogous to that of the corollary to Theorem 2.2.

Example 2.2. For our two-body jumper, the necessary condition for collision-free landing means that at the moment the leg touches the support its velocity must vanish, while the height of the body over the support is $q_2^J=a+m_1gk^{-1}$. The sufficient conditions (Theorem 2.4) mean that, in addition, the velocity $q_2^\bullet$ of the body is negative.

3. CONSTRUCTION OF MOTIONS WITH COLLISION-FREE FLIGHTS

We will now look for motions of system (1.1) that include segments of different types, alternating according to the scheme 2–3–1–5*a*–2. A rough representation of the graph of $q_1(t)$ in such motions is shown in Fig. 4, we shall call segment 1 in flight between two strained intervals of type 2.

By Theorem 2.1, the slackening must occur at a point of J; the subsequent type 1 motion is described by the formulas

$$\mathbf{q}=\mathbf{q}\,(t,\mathbf{q}^\circ,\mathbf{q}^{\bullet\circ}) \tag{3.1}$$

where $\mathbf{q}^\circ$ and $\mathbf{q}^{\bullet\circ}$ are the values of the variables at the instant $t=t^\circ$ the constraint is slackened. The functions (3.1) are a solution of system (1.1), where $R\equiv 0$.

At times $t>t^\circ$ sufficiently close to t°, the value of q_1 in (3.1) is positive. Let us assume that $q_1>0$ for $t^\circ<t<T$ and $q_1(T)=0$.

We define a mapping $\mathbf{\Phi}$ of the part of the J in which $q_1=q_1^\bullet=0$ into the plane $q_1=0$, as follows:

$$\mathbf{\Phi}\,(\mathbf{u}^\circ)=\mathbf{u}\,(T),\quad \mathbf{u}=(\mathbf{q},\mathbf{q}^\bullet),\quad \mathbf{u}^\circ=(\mathbf{q}^\circ,\mathbf{q}^{\bullet\circ}) \tag{3.2}$$

This mapping $\mathbf{\Phi}$ may also be defined implicitly by formulas (3.1), by setting $t=T$ and adding the equality

$$q_1\,(T,\mathbf{q}^\circ,\mathbf{q}^{\bullet\circ})=0 \tag{3.3}$$

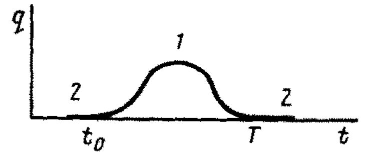

FIG. 4.

Depending on the nature of the strained state of the constraint, $\mathbf{\Phi}$ may have different analytical properties, as we shall show below.

Theorem 3.1. 1. $\mathbf{\Phi}$ is differentiable at points corresponding to trajectories for which straining of the constraint is accompanied by a collision (cases 4b, 4c or 5b in Fig. 1).

2. $\mathbf{\Phi}$ is continuous but not differentiable at points corresponding to collision-free straining of the constraint (case 5a), following which the representative point enters the domain J^-.

3. If $\mathbf{u}(t) \in J^+ \cup J$ for $t > T$ (as in case 4a), the $\mathbf{\Phi}$ is discontinuous at the given point $\mathbf{u}^\circ$.

Proof. 1. In this case

$$\frac{\partial q_1}{\partial t}(t, \mathbf{u}^\circ)\Big|_{t=T} = q_1^{\boldsymbol{\cdot}}(T) < 0$$

and by the Implicit Function Theorem Eq. (3.3) defines a differentiable function $T = T(\mathbf{u}^\circ)$ in the neighbourhood of the given point $\mathbf{u}^\circ$. Substituting T for t in (3.1), we obtain the explicit form of the differentiable mapping $\mathbf{\Phi}$.

2. Since $\mathbf{u}(t) \in J^-$ for $t > T$, it follows that then $q_1(t, \mathbf{u}^\circ) < 0$ in (3.1). Since the functions (3.1) are continuous, the curve $\mathbf{u}(t, \mathbf{u}^\circ + \Delta\mathbf{u}^\circ)$ will also intersect the plane $q_1 = 0$, and it will do so at a point near $\mathbf{\Phi}(\mathbf{u}^\circ)$ but this means that $\mathbf{\Phi}$ is continuous.

The fact that the function is not differentiable follows from Lindelöf's theorem as to the dependence of the solution of a system

$$\mathbf{u} = \mathbf{F}(\mathbf{u}), \quad \mathbf{u}(t^\circ) = \mathbf{u}^\circ$$

on the initial data [5]: for any $t > t^\circ$ the Jacobian $|\partial\mathbf{u}/\partial\mathbf{u}^\circ|$ is non-zero. Since

$$\frac{\partial q_1}{\partial t}(t, \mathbf{u}^\circ)\Big|_{t=T} = 0$$

at least one of the quotients $\Delta T/\Delta u_j^\circ$ $(j = 1, \ldots, 2n)$ is unbounded in the neighbourhood of $\mathbf{u}^\circ$.

Since $\mathbf{\Phi}(\mathbf{u}^\circ)$ is not an equilibrium position, at least one of the derivatives $\partial u_s/\partial t (s = 1, \ldots, 2n)$ is not zero at $t = T$. Thus the partial derivative $\partial u_s/\partial u_j^\circ$ does not exist.

3. In this case the phase curve is described by formulas (3.1) not only in the interval $t^\circ < t < T$ but also for $t > T$, since R vanishes in the domain $J \cup J^+$. The vector field $\mathbf{F}(\mathbf{u})$ in the neighbourhood of a non-singular point $\mathbf{\Phi}(\mathbf{u}^\circ)$ is diffeomorphic to a constant vector field [5], and under the diffeomorphism the plane $q_1 = 0$ is mapped onto some smooth surface. Since the unperturbed curve $\mathbf{u}(t, \mathbf{u}^\circ)$ touches this surface but does not cut it, while the curves $\mathbf{u}(t, \mathbf{u}^\circ + \Delta\mathbf{u}^\circ)$ are parallel to it, some of these curves have no points in common with the surface in a neighbourhood of $t = T$. Consequently, $\Delta\mathbf{\Phi} \not\to 0$ as $\Delta\mathbf{u}^\circ \to 0$.

This theorem, combined with the results of Sec. 2, yields an algorithm to construct motions with collision-free flights: by analytical or numerical means, one finds points at which the range of the mapping $\mathbf{\Phi}$ intersects the plane $q_1 = q_1^{\boldsymbol{\cdot}} = 0$ and the surface J, and then verifies the conditions of the Theorem 2.4.

Example 3.1. Formulas (3.1) are easily worked out for the two-body jumper, by solving the linear system (2.4) for $R = 0$:

$$q_1 = g\omega^{-2}(1 - \cos\nu - {}^1\!/_2\nu^2 + \alpha\nu - \alpha\sin\nu) \tag{3.4}$$
$$q_2 = q_2^\circ + g\omega^{-2}[\alpha\nu - {}^1\!/_2\nu^2 + (m_1/m_2)(\alpha\sin\nu + \cos\nu - 1)]$$
$$\nu = \omega t, \quad \alpha = \frac{\omega m_2}{Mg} q_2^{\boldsymbol{\cdot}\circ}, \quad M = m_1 + m_2, \quad \omega^2 = \frac{kM}{m_1 m_2}$$

The necessary conditions for a collision-free landing, $q_1 = q_1^{\boldsymbol{\cdot}} = q_1^{\boldsymbol{\cdot\cdot}} = 0$, taking (3.4) into consideration, may be written in the form

$$\alpha(\nu - \sin\nu) = \cos\nu - 1 + {}^1/_2\nu^2, \quad \alpha(1 - \cos\nu) = \nu - \sin\nu, \quad \alpha\sin\nu = 1 - \cos\nu \tag{3.5}$$

The solution of system (3.5) are described by the formulas

$$\alpha = \operatorname{tg}\alpha, \quad \nu = 2\alpha \tag{3.6}$$

The sufficient conditions, besides (3.6), reduce to the inequality $q_2^{\cdot} < 0$, whence, using (3.4), we obtain $\alpha > 0$. Equation (3.6) has an infinite number of positive roots, the smallest of which lies in the interval $(\pi, {}^3\!/_2\pi)$. For each root we have a collision-free flight between the point

$$q_2^{\circ} = q_2^{J}, \quad q_2^{\cdot\circ} = \frac{Mg}{\omega m_2}\alpha \tag{3.7}$$

and its mirror image with respect to the q_2 axis.

To express ν as a function of α at the end of the flight, we use the first of formulas (3.5), which gives

$$\alpha = \frac{\cos\nu - 1 + {}^1/_2\nu^2}{\nu - \sin\nu}, \quad \frac{d\alpha}{d\nu} = \frac{(\nu\cos{}^1/_2\nu - 2\sin{}^1/_2\nu)^2}{(\nu - \sin\nu)^2} \geqslant 0 \tag{3.8}$$

It is obvious from (3.8) that $\nu(\alpha)$ is a continuous and monotone increasing function for $\alpha > 0$, but it is non-differentiable at points where $\nu = 2\operatorname{tg}{}^1\!/_2\nu$, in which case $\alpha = \operatorname{tg}{}^1\!/_2\nu = {}^1\!/_2\nu$. Thus, in keeping with the conclusions of Theorem 3.1, loss of differentiability is always associated with collision-free motions (3.6).

4. PERIODIC COLLISION-FREE MOTIONS AND THEIR ULTIMATE PROPERTIES

If the forces acting on the system are conservative, the mechanical energy of the system will be conserved over intervals of collision-free motion. However, if collisions occur, and they are not absolutely elastic [$\varkappa < 1$ in (1.3)], then energy will be dissipated and the system will not be conservative. Consequently, if periodic motions exist, they are collision-free.

If the constraint remains slackened throughout some periodic motion, $q_1 > 0$, it will have no influence on other motions sufficiently close to the periodic motion, so that we can employ perturbation methods to investigate stability.

If the periodic motions contain intervals over which the constraint is strained ($q_1 = 0$), the perturbed motion will generally be accompanied by collisions with the constraint. This case, therefore, will need special study. Some questions of stability in systems with absolutely elastic collisions ($\varkappa = 1$) were considered in [6, 7]. In this paper we will discuss the case $\varkappa < 1$ in a system with two degrees of freedom.

We shall assume that the Lagrangian has the form

$$L = {}^1/_2(a_{11}q_1^{\cdot 2} + a_{22}q_2^{\cdot 2}) + a_1q_1^{\cdot} + a_2q_2^{\cdot} + L_0(\mathbf{q}), \quad q_1 \geqslant 0 \tag{4.1}$$

where the coefficients depend on $\mathbf{q}$, and $Q_1 = Q_2 \equiv 0$. The absence of a term $a_{12}q_1^{\cdot}q_2^{\cdot}$ does not affect the generality of the argument, only requiring a special choice of the generalized coordinates [8]. If the quadratic part of the Lagrangian in some cooordinate system q_1^*, q_2^* is

$$L_2^* = {}^1/_2(a_{11}^*q_1^{\cdot *2} + 2a_{12}^*q_1^{\cdot *}q_2^{\cdot *} + a_{22}^*q_2^{\cdot *2})$$

then the following change of variables will produce the form (4.1) [8]:

$$q_1 = q_1^*, \; q_2^* = \varphi(q_1, q_2), \; \partial\varphi/\partial q_1 = -a_{12}^*/a_{22}^*, \; \varphi(0, q_2) = q_2 \tag{4.2}$$

The existence of an invertible change of variable (4.2) for all $q_1 > 0$ follows, once again, from Lindelöf's theorem; the explicit form of the substitution is unimportant in applications, since the results of the qualitative analysis are invariant with respect to the choice of coordinates.

The distinctive feature of the equations of motion of system (4.1) is that $q_2^{\cdot\cdot}$ is independent of the reaction of the constraint.

Over intervals of collision-free motion, the phase curves are confined to surfaces of constant mechanical energy:

$$h = L_2 - L_0 = \text{const} \tag{4.3}$$

The strained intervals of the constraint may be represented in the phase plane $(q_2, q_2^{\cdot})$ by tracing out the curves (4.3) for $q_1 = q_1^{\cdot} = 0$. If any such curve is closed, lies entirely within J^- and contains no equilibrium positions of the system, it will describe a periodic motion in which the constraint remains strained at all times (curve 1 in Fig. 3a represents this type). The same is true of curves corresponding to sufficiently close values of the mechanical energy h.

A second type of periodic motion is described by curves (4.3) that have points in common with the surface J but do not intersect J^+ (curve 2 in Fig. 3a).

Finally, a third type of periodic motion is obtained for curves (4.3) that intersect the surface J (curve 3 in Fig. 3a). In that case the representative point moves along the curve only as long as it remains in the domain $J^- \cup J$. On crossing J the phase curve leaves the plane $q_1 = 0$, so that it is not possible to represent the next interval of motion in the $(q_2, q_2^{\cdot})$ plane. Upon collision-free renewal of contact, the representative point returns to the plane $q_1 = q_1^{\cdot} = 0$, while remaining on the same surface (4.3); by Theorem 2.4, it will then cross J. We note that periodic motions of this type may exist even if Eq. (4.3) does not describe a closed curve in the $(q_2, q_2^{\cdot})$ plane; the only requirement is that the curve should connect the point at which contact is renewed with the next point at which contact is broken.

We will now prove two auxiliary propositions.

Lemma 4.1. Let $\mathbf{q}(t)$, $\mathbf{x}(t)$ be vector-valued functions that satisfy the following systems of second-order differential equations:

$$\begin{gathered} \mathbf{q}^{\cdot\cdot} = \mathbf{F}(\mathbf{q}, \mathbf{q}^{\cdot}), \quad \mathbf{x}^{\cdot\cdot} = \mathbf{F}(0, x_2, 0, x_2^{\cdot}) \\ \mathbf{x}(t^\circ) = \mathbf{q}(t^\circ) = \mathbf{q}^\circ, \quad \mathbf{x}^{\cdot}(t^\circ) = \mathbf{q}^{\cdot}(t^\circ) = \mathbf{q}^{\cdot\circ} \\ \mathbf{x}, \mathbf{q} \in R^2, \quad \mathbf{F} \in c_1(\Omega), \quad t^\circ \leqslant t \leqslant t^\circ + \Delta t \end{gathered} \tag{4.4}$$

where $\Omega \subset R^4$ is a bounded closed domain.

Then

$$\begin{gathered} \|\mathbf{q} - \mathbf{x}\| = o(\|(x_1, 0)\|) \quad \text{as} \quad \Delta t \to 0 \\ \|\mathbf{x}\| = \max_t [\mathbf{x}^2(t) + \mathbf{x}^{\cdot 2}(t)]^{1/2} \end{gathered} \tag{4.5}$$

Proof. Transform Eqs (4.4) to integral form and subtract one from the other:

$$\begin{gathered} \mathbf{q}(t) - \mathbf{x}(t) = \mathbf{A}\mathbf{q}(t) \\ \mathbf{A}\mathbf{q}(t) = \int_{t^\circ}^{t} d\tau \int_{t^\circ}^{\tau} [\mathbf{F}(\mathbf{q}(s), \mathbf{q}^{\cdot}(s)) - \mathbf{F}(0, x_2(s), 0, x_2^{\cdot}(s))]\, ds \end{gathered} \tag{4.6}$$

Treating $\mathbf{x}(t)$ in (4.6) as a known function of time, we can determine $\mathbf{q}(t)$ by successive approximations. Setting

$$\begin{gathered} \mathbf{q}^\circ(t) = (0, x_2(t)), \quad \mathbf{q}^m(t) = \mathbf{x}(t) + \mathbf{A}\mathbf{q}^{m-1}(t) \\ m = 1, 2, \ldots, \quad \mathbf{q}(t) = \lim_{m\to\infty} \mathbf{q}^m(t) \end{gathered} \tag{4.7}$$

we can construct a solution of Eq. (4.6) on the assumption that $\mathbf{A}$ is a contractive mapping [9]. Since

$$\|\mathbf{A}\mathbf{q}' - \mathbf{A}\mathbf{q}''\| \leqslant C(\Delta t)^2 \|\mathbf{q}' - \mathbf{q}''\|, \quad C = \max_\Omega \|d\mathbf{F}\|$$

the mapping will indeed be contractive, provided $\lambda = C(\Delta t)^2$ is less than 1.

Noting that by (4.6) and (4.7) $\mathbf{q}^1 = \mathbf{x}$, we obtain

$$\|\mathbf{q} - \mathbf{x}\| = \|\mathbf{q} - \mathbf{q}^1\| = \|\mathbf{A}\mathbf{q} - \mathbf{A}\mathbf{q}^\circ\| \leqslant \lambda \|\mathbf{q} - \mathbf{q}^\circ\| \leqslant \lambda (\|\mathbf{q} - \mathbf{x}\| + \|\mathbf{q}^1 - \mathbf{q}^\circ\|)$$

whence, since ($\lambda = o(\Delta t)$), the estimate (4.5) now follows.

We will use Lemma 4.1 to investigate the equations of motion of system (4.1). The auxiliary system defined in (4.4) is in that case

$$x_1^{\cdot\cdot} = a_{11}^{-1}\,(\partial L/\partial x_1 - a_1^{\cdot})\,|_{x_1=x_1^{\cdot}=0} \qquad (4.8)$$
$$x_2^{\cdot\cdot} = a_{22}^{-1}\,(\partial L/\partial x_2 - a_{22}^{\cdot}x_2^{\cdot} - a_2^{\cdot})\,|_{x_1=x_1^{\cdot}=0}$$

The second of these equations defines $x_2(t)$. Once it is solved, we can also solve the first equation:

$$x_1^{\cdot\cdot} = f(t), \quad x_1^{\cdot} = q_1^{\cdot\circ} + \int_{t^\circ}^{t} f(t)\,dt \qquad (4.9)$$
$$x_1 = q_1^\circ + (t - t^\circ)\,q_1^{\cdot\circ} + \int_{t^\circ}^{t} du \int_{t^\circ}^{u} f(s)\,ds$$

Lemma 4.2. Assume that $y(t)$ satisfies the equation

$$y^{\cdot\cdot} = f(t), \quad y(0) = 0, \quad y^{\cdot}(0) = O(\delta) \quad \text{as} \quad \delta \to 0, \ f \in C_1(R)$$

and let τ be a time interval throughout which $y>0$. Then the following estimates hold:

1. If $f(0) = -c<0$, then $\tau = O(\delta)$; $y = O(\delta^2)$, $y^{\cdot} = O(\delta)$ for $0 \leqslant t \leqslant \tau$ and $y^{\cdot}(\tau) = -y^{\cdot}(0) + O(y^{\cdot 2}(0))$.
2. If $f(0) = O(\delta^{1/2}) \geqslant 0, f'(0) = -c_1<0$, then $\tau = O(\delta^{1/2}), f(\tau) = O(\delta^{1/2})<0, y^{\cdot}(\tau) = O(\delta)$.
3. If $f(0) = O(\delta^{1/2})<0, f'(0) = -c_1<0$, then $\tau = O(\delta^{1/2}), y^{\cdot}(\tau) = -2y^{\cdot}(0) - \tfrac{1}{2}f(0)\tau + o(\delta)$.

Proof. 1. In this case

$$y^{\cdot\cdot} = -c + O(\tau), \ y^{\cdot} = y^{\cdot}(0) - c\tau + O(\tau^2), \ y = \tau\,(y^{\cdot}(0) - {}^1\!/_2 c\tau + O(\tau^2))$$

Equating y to zero we obtain $\tau = 2\delta c^{-2} + o(\delta)$, implying the other assertions.

2. In this case relations (4.9) become

$$y^{\cdot\cdot} = f(0) - c_1\tau + O(\tau^2), \ y^{\cdot} = y^{\cdot}(0) + \tau f(0) - {}^1\!/_2 c_1\tau^2 + O(\tau^3) \qquad (4.10)$$
$$y = \tau\,(y^{\cdot}(0) + {}^1\!/_2\tau f(0) - {}^1\!/_6 c_1\tau^2 + O(\tau^3))$$

We determine τ from the equation

$${}^1\!/_6 c_1\tau^2 - {}^1\!/_2 f(0)\,\tau - y^{\cdot}(0) + O(\tau^3) = 0$$

from which it follows that $\tau = O(\delta^{1/2}), y^{\cdot}(\tau) = -2y^{\cdot}(0) - \tfrac{1}{2}\tau f(0) + O(\tau^2)$ and so on.

3. Here again formulas (4.10) hold, but now with $f(0)<0$. This implies the desired estimates and the inequality

$$y^{\cdot}(\tau) + 2y^{\cdot}(0) > 0 \qquad (4.11)$$

Let us consider the first type of periodic motion.

Theorem 4.1. If a periodic motion is described by a closed curve Π situated entirely within J^- in the plane $q_1 = q_1^{\cdot} = 0$, then it is orbitally stable.

Proof. The assertion means that for any positive number ε there exists $\delta = \delta(\varepsilon)>0$ such that, if at some time $t = t^\circ$ the distance between $\mathbf{u}^\circ$ and Π is less than δ, then the phase curve through the point is distant less than ε from Π for all $t>t^\circ$.

Since $\Pi \subset J^-$, it follows that also $\mathbf{u}^\circ \in J^-$ for sufficiently small δ, so that at some time in an interval of the order of δ the phase curve will intersect the plane $q_1 = 0$. If $\varkappa = 0$ the representative point will remain in the same plane (case 5b), moving along one of the closed curves (4.3), which will merge with Π as $\delta \to 0$.

If $0<\varkappa<1$, a collision at $t = t_1$ will lead to subsequent slackening of the constraint (case 4b) and repeated collisions. Let v_k be the value of $q_1^{\cdot}$ after the kth collision. Then, applying Lemma 4.2

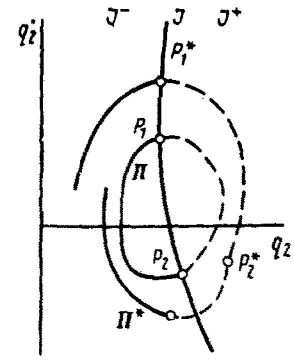

FIG. 5.

(case 1), we see that before the $(k+1)$th collision the quantity $y^\circ = x_1^\circ$ will be equal to $-v_k + O(v_k^2)$. Hence, by Lemma 4.1, $v_{k+1} = \varkappa v_k + O(v_k^2)$, and for sufficiently small initial perturbations δ we obtain

$$v_{k+1} \leqslant \varkappa' v_k, \quad \varkappa \leqslant \varkappa' < 1 \tag{4.12}$$

Consequently, the repeated collisions will be damped out over a time of the order of δ (see [4]), and when that happens the total energy dissipation will also tend to zero together with δ. After the collisions have been damped out the representative point will lie on a closed curve Π^* in the plane $q_1 = q_1^\cdot = 0$, which will merge with Π as $\delta \to 0$.

We now turn to the third type of periodic collision-free motion. Let Π be a trajectory made up of two parts:

(a) the part of the curve between points P_1 and P_2 is described in the phase plane $(q_2, q_2^\cdot)$ by (4.3);

(b) the part of the curve between P_1 and P_2 is not in the plane $q = q_1^\cdot = 0$; it is described by formulas

$$\mathbf{u} = \mathbf{u}(h, t), \ \mathbf{u}(h, 0) = P_1, \ \mathbf{u}(h, T) = P_2, \ \mathbf{u} = (\mathbf{q}, \mathbf{q}^\cdot) \tag{4.13}$$

(see Fig. 5). By Theorems 2.2 and 2.4, P_1 and P_2 must lie on the surface of separation J.

Theorem 4.2. Assume that the following conditions are satisfied:

1. The curve Π intersects the surface at the points P_1 and P_2 at a non-zero angle.
2. $\partial q_1(h, T)/\partial h \neq 0$ in formula (4.13).
3. The coefficient of restitution $\varkappa$ is less than ½.

Then Π is semistable, that is, it lies on the boundary of domains of attraction and repulsion in the phase space R^4.

Proof. Under the assumptions of the theorem, the trajectory of perturbed motion Π^* will cut the plane $q_1 = 0$ at some point P_2^* near P_2 (it is not necessarily true that $P_2^* \in J$; see Fig. 5). The motion prior to the collision is described by the first of formulas (4.13). As follows from Theorem 2.4 and condition 2, we can rewrite the equation $q_1(h, T) = 0$ in the neighbourhood of P_2 in the form

$$q_1(h, T) = b\Delta h + e(\Delta T)^3 + o(|\Delta h| + |\Delta T|^3) = 0 \tag{4.14}$$
$$b = \frac{\partial q_1(h, T)}{\partial h} \neq 0, \quad e = \frac{1}{6}\frac{\partial^3 q_1(h, T)}{\partial t^3} < 0$$

Solving these equations, we obtain an estimate

$$\Delta T = -b^{1/3} e^{-1/3} \mu + o(\mu), \quad \mu = (h^* - h)^{1/3} \tag{4.15}$$

where h^* is the energy of the perturbed system over the "flight" interval. Differentiating (4.14) with respect to time and using (4.15), we obtain

$$q_1^{\cdot}(h, T) = 3e(\Delta T)^2 + o(\Delta T)^2 = 3b^{2/3}e^{-1/3}\mu^2 + o(\mu^2) \qquad (4.16)$$

$$\Delta h_1 = {}^1\!/_2 (1 - \varkappa^2)\, a_{11} q_1^{\cdot 2}(h, T) = {}^9\!/_2 (1 - \varkappa^2)\, a_{11} b^{4/3} e^{2/3} \mu^4 + o(\mu^4)$$

where Δh_1 is the energy dissipated in the collision at P_2^*.

The motion of the representative point after collision will depend on the value of $\varkappa$ and also on the position of P_2^* relative to the surface J. If $\varkappa = 0$, $P_2^* \in J^- \cup J$, then after collision (case 5b) the representative point will continue to move in the plane $q_1 = q_1^{\cdot} = 0$ until the next intersection with J in the neighbourhood of P_1; there will then be a flight, followed by a collision at some point P_2^{**} in the neighbourhood of P_2; and so on. The above-mentioned domains of attraction and repulsion are defined in the neighbourhood of Π by the inequalities

$$h^* > h + \Delta h_1, \quad h^* < h \qquad (4.17)$$

where Δh_1 is the quantity defined in (4.16); to a first approximation, the attraction condition is

$$h < h^* < h + \frac{8}{729} a_{11}^{-3} b^{-4} e^{-2} \qquad (4.18)$$

If $\varkappa = 0$, $P_2^* \in J^+$, the representative point will leave the phase plane $(q_2, q_2^{\cdot})$ after collision (case 4c); since motion along Π takes place at non-zero velocity towards the domain J^-, we can apply the second assertion of Lemma 4.2. The result is that the system experiences a flight of duration $\tau = O(\delta^{1/2}) = O(\mu)$, at the end of which $y^{\cdot} = x_1^{\cdot}$ will be a quantity of the order of μ^2. By Lemma 4.1, the same is true of $q_1^{\cdot}$ and the representative point will enter J^-.

The energy Δh_2 dissipated in the second collision will be of the order of μ^4, and then the motion will continue in the $(q_2, q_2^{\cdot})$ plane. Consequently, in this case too the domain of attraction defined by the inequality

$$h^* > h + \Delta h_1 + \Delta h_2$$

is not empty.

The case $\varkappa \in (0, 1/2)$ is treated similarly. Here the perturbed trajectory Π^* will be made up of sections in the phase plane $(q_2, q_2^{\cdot})$, alternating with flights when the surface J is crossed, and series of intervals over which the collisions are repeatedly damped out in the neighbourhood of P_2. That the collisions will be damped out over a time interval of the order of δ is proved just as in Theorem 4.3; the damping condition (4.12) for $\varkappa < 1/2$ follows in this case from (4.11). The total energy dissipation in a quasi-static collision

$$\Delta h_\Sigma = \Delta h_1 + \Delta h_2 + \ldots$$

is a quantity of the order of μ^4, so that the domain of attraction

$$h^* > h + \Delta h_\Sigma$$

is again not empty; the domain of repulsion is described by the second inequality in (4.17).

Example 4.1. Periodic collision-free motions of the first type for our two-body jumper are described by the curves (2.6) that lie entirely in the strip $0 < q_2 < q_2^J$ ($q_2 > 0$ by physical considerations). By Theorem 4.3, these motions are orbitally stable.

Motions of the third type are described by conditions (3.6); remembering that always $q_2 > 0$, we conclude that there are only a finite number of such motions, the actual number depending on the parameters of the system. For $h^* > h$ we have $\Delta\alpha > 0$, as follows from (3.4), and in that case the point P_2^* is in J^-. The domain of attraction in the case $\varkappa = 0$ is defined by (4.18), so that, using (3.4)–(3.8), we get

$$\Delta\alpha < \frac{2M^6}{81\alpha^3 m_1^3 m_2^3 \sin^3\alpha}$$

(the part of the domain of attraction lying in the phase space is shown hatched in Fig. 3b).

Example 4.2. The Lagrangian of a link-up of two bodies in a satellite orbit is [2]

$$L = {}^1/_2\,(r^{\cdot 2} + r^2\varphi^{\cdot 2}) - r^2\varphi^{\cdot} + {}^3/_2 r^2 \sin^2\varphi, \quad r \leqslant 1$$

and the surface of separation is defined by the equation

$$\varphi^{\cdot 2} - 2\varphi^{\cdot} + 3\sin^2\varphi = 0$$

(replacing the equality sign here by > defines the domain J^-).

Formulas (3.1), which describe the flight phase, are

$$\eta = r\sin\varphi = B_0 + B_2\cos t + B_3\sin t$$
$$\zeta = r\cos\varphi = A_0 - {}^3/_2 B_0 t + 2B_3\cos t - 2B_2\sin t$$
$$A_0 = \zeta_0 - 2\eta_0^{\cdot},\ B_0 = 2\,(2\eta_0 + \zeta_0^{\cdot}),\quad B_2 = -3\eta_0 - 2\zeta_0^{\cdot},\quad B_3 = \eta_0^{\cdot}$$

Periodic motions with collision-free flights may be constructed by setting

$$\eta\,(T) = \eta_0,\quad \zeta\,(T) = -\zeta_0,\quad \eta^{\cdot}\,(T) = -\eta_0^{\cdot},\quad \zeta^{\cdot}\,(T) = \zeta_0^{\cdot}$$

and hence we obtain the conditions

$$B_3\cos{}^1/_2 t = B_2\sin{}^1/_2 t,\quad A_0 = {}^3/_4 B_0 t \tag{4.19}$$

Equations (4.19) have an infinite number of solutions; in the terminology of this paper, they define periodic motions of the third type. These motions were also obtained numerically in [2], on the assumption that $\varkappa = 0$. The final conclusion was that these motions actually occur for a set of initial data of measure zero only.

The results of Theorem 4.2 indicate that when $0 \leqslant \varkappa \leqslant ½$ each of the periodic motions has a domain of attraction. The fact that these domains were not detected by numerical analysis may be attributed to inadequate computational accuracy.

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